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Calculation of boundary-layer development using the turbulent energy equation: compressible flow on adiabatic walls

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The basic method described by Bradshaw, Ferriss & Atwell (1967) is extended to compressible flow in two-dimensional boundary layers in arbitrary pressure gradient (excluding shock waves and expansion fans) by invoking Morkovin's hypothesis (Favre 1964) that the turbulence structure is unaffected by compressibility. Using the same empirical functions as in incompressible flow, skin friction in zero pressure gradient is predicted to within 3 % of Spalding & Chi's (1964) correlation for free-stream Mach numbers less than 5. Comparisons with experiments in pressure gradient are restricted by the lack of data, but, since Morkovin's hypothesis does not depend on pressure gradient, methods which use it (of which the present method seems to be the first) can be checked fairly adequately by comparisons with data in zero pressure gradient.

No 'compressibility transformations' are needed, although the Crocco relation is used, provisionally, for the temperature: since the calculations take only about 20 % longer than in incompressible flow, Morkovin's hypothesis does as much as any transformation could do. It is pointed out that, in supersonic flow, surface curvature which is large enough to induce a significant longitudinal pressure gradient is also large enough to have a very significant effect on the turbulence structure.

1. Introduction

Bradshaw *et al.* (1967, hereafter cited as Q) described a method of boundary-layer calculation, based on the turbulent energy equation for $D\frac{1}{2}\overline{q^2}/Dt$, the rate of change of $\frac{1}{2}\overline{q^2} \equiv \frac{1}{2}(\overline{u^2}+\overline{v^2}+\overline{w^2})$ along a mean streamline (Townsend 1956, equation 2.4.10). Most previous calculation methods relied on assumed relations between properties of the turbulence (such as shear stress or entrainment) and properties of the mean velocity field (such as velocity gradient or profile shape parameter). The method of Q involves only assumptions about the relations between one turbulence property and another—that is, about the turbulence structure. ('Structure' is used here in the sense of Townsend (1956) and many other authors, as a collective noun for the relations between the statistical properties of the turbulence.) The basic assumption of Q was that the turbulence structure at a given streamwise position is uniquely specified by the turbulent

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shear-stress profile so that, for instance, for a given shear-stress profile there is one and only one intensity profile, the relation between the two being derived from experimental data. By introducing empirical relations of this sort between the shear stress and all the other turbulence properties appearing in the (exact) turbulent energy equation, that equation can be converted into an (approximate) equation for $D(-\overline{uv})/Dt$, the rate of change of turbulent shear stress along a mean streamline. Rather simple empirical relations give results of good engineering accuracy as long as the boundary-layer approximation is obeyed (Kline *et al.* 1969). In principle, the same basic assumption about the turbulence structure could be used directly to convert the exact equation for $D(-\overline{uv})/Dt$ (Townsend 1956, equation 2.4.9) into a closed (approximate) form, but this equation contains correlations between the fluctuations of velocity and pressure which cannot at present be measured or estimated with any certainty.

Because the method of Q relies solely on assumptions about the turbulence structure, it can be extended quite easily to compressible flow by invoking a simple hypothesis first clearly formulated by Morkovin in 1961 (published in Favre 1964). Morkovin pointed out that one would not expect the turbulence structure (specifically, dimensionless quantities like anisotropy parameters, spectrum shapes and the like) to be affected by compressibility as long as the Mach number fluctuation is much less than unity (so that density fluctuations are small compared to the mean density). This condition is satisfied by all non-hypersonic boundary layers (free-stream Mach number M_1 less than 5, say). Morkovin's hypothesis is supported by his analysis of the measurements of Kistler (1959), by the later measurements of Demetriades (1968) and *a posteriori* by the success of the calculation method described below. It is much more subtle, and more plausible, than the hypothesis that 'eddy viscosity' relations between the shear stress and the mean flow are unaltered by compressibility. The mean flow is greatly altered by compressibility, particularly in a longitudinal pressure gradient, and nothing that we know about turbulence supports the idea that the shear stress should be altered in just the same way: assumptions of this sort can only be justified by direct appeal to experiments in compressible flow and it is unfortunately the case that there are literally no compressible boundary-layer measurements, except in zero pressure gradient, that can be used as reliable and severe test cases for calculation methods (this melancholy statement will be elaborated in § 7). It follows that prediction of compressible boundary layers to the high accuracy needed by the aircraft industry can only be achieved at present by extensions of incompressible-flow methods, using additional hypotheses that can be adequately tested in zero pressure gradient. Morkovin's hypothesis is independent of pressure gradient, except that one must always expect changes in turbulence structure if rates of strain other than the mean shear become appreciable, so that the hypothesis would fail in the presence of shock waves and expansion fans where the dilatation div $\mathbf{U}$ is large. Therefore *if* Morkovin's hypothesis is applied to an incompressible-flow method that gives good results in a wide range of pressure gradients *and* the resulting compressible-flow method gives good results in zero pressure gradient *then* (i) Morkovin's hypothesis is justified; (ii) the compressible-flow method will give good results in a wide range of pressure gradients not involving shock waves, expansion fans or other violations of the boundary-layer approximation.

In the present paper this piece of logic is applied to the method of Q.

Another method of proceeding from incompressible to compressible flow is via a density transformation of the sort rigorously derived for laminar boundary layers and since unrigorously applied to turbulent boundary layers. Our position on transformations is the same as that of McDonald (Bertram 1969): "In summary it does not seem that the presently available transformations can withstand a careful scrutiny" (see also Bradshaw, Sivasegaram & Whitelaw 1970). To be reliable, a transformation must, surely, either deal explicitly with the turbulence as well as the mean flow (so that its physical plausibility can be assessed) or be accompanied by an existence proof. Since this has not yet been accomplished, even for zero pressure gradient, the prospects of a general transformation capable of competing directly with Morkovin's hypothesis seem remote.

The use made of Morkovin's hypothesis in the present paper to extend the method of Q to compressible flow is straightforward: we simply assume that the empirical relations between the shear stress and the other turbulence properties used in the method of Q are unaltered by compressibility (but see the beginning of § 4 for discussion of a special point). Having made this assumption, no further physical input is necessary (except that, as in incompressible flow, we use the universal inner-law velocity profile as the inner boundary condition). Since the method of Q has been tested as well as the incompressible data allow (see Kline *et al.* 1969) there is no point in further trial-and-error adjustment of the empirical functions to improve agreement with the data. Furthermore, the general behaviour of the compressible version at non-hypersonic Mach numbers—for example its sensitivity to initial conditions or to changes in the empirical relations—is bound to be qualitatively the same as that of the incompressible method. From the point of view of the student of compressible turbulent flow, Morkovin's hypothesis spoils the fun; but it is a great comfort to the user of calculation methods. This does not exclude difficulties from phenomena like shock waves which are peculiar to supersonic flow or from special influences like surface curvature and low Reynolds number which are more important in supersonic flow (see §§ 4 and 6).

In this paper we describe the extension of the method of Q to compressible boundary layers with zero heat transfer at the surface: by accepting the restriction that $(\gamma-1)M^2$ shall not be an order of magnitude greater than unity (a stronger restriction than Morkovin's) the effects of density fluctuations, though not of mean density variations, can be virtually eliminated from the equations. This limits the free-stream Mach number to about 3 (in fact the results at $M_1 = 5$ in zero pressure gradient are still satisfactory) but above this Mach number, boundary layers with zero heat transfer are of little practical importance. So far, the main use of the present method in industry seems to have been in high subsonic and transonic flow. The restriction to zero heat transfer is not insurmountable, but we considered it advisable initially to develop the heat-transfer version of the method in incompressible flow with small temperature differences (Bradshaw & Ferriss 1968): the extension to compressible heat transfer is now running with satisfactory results (Bradshaw & Ferriss 1970) and further numerical work is in progress. In the present calculations the temperature distribution across the boundary layer is assumed to be given by the Crocco relation

$$c_p T + \tfrac{1}{2} r U^2 = \text{constant} = c_p T_w, \tag{1}$$

with $r = 0{\cdot}89$, which is known to be a good approximation, although it does not satisfy the thermal energy equation

$$\int_0^\delta \rho U (T_0 - T_{01})\, dy = 0$$

for an adiabatic wall because the small positive values of $T_0 - T_{01}$ in the outer part of the boundary layer are neglected. (Here suffixes 0, 1 and w denote stagnation, free-stream and wall conditions respectively.) It appears from experimental data that the recovery factor does not depend greatly on pressure gradient. The effect of taking the recovery factor to be *unity* instead of about $0{\cdot}9$ is to decrease the skin friction in zero pressure gradient by about 5 % at $M_1 = 2{\cdot}2$: clearly uncertainties of 1 or 2 % in the recovery factor will not affect velocity calculations, although they affect integral quantities via the density profile. To put the Crocco relation in context we may note that it is frequently used (even with $r = 1$) to reduce experimental data, so that it is accurate to within the likely error of velocity profile data. The Crocco relation, with the addition of a linear term, can be used in principle in boundary layers with heat transfer (see, for instance, Rotta 1965) but it does not seem to agree well with experiment and we have not attempted to use it for heat-transfer calculations. In adiabatic flow it is a great numerical simplification.

It is difficult to compare the present assumption about the effect of compressibility on turbulence with the assumptions made by previous authors, because in no case known to us have previous authors explicitly invoked Morkovin's hypothesis. Since Morkovin's hypothesis is an obvious point of reference for compressible calculation methods it is surprising to find it generally ignored by authors of calculation methods and of review articles.

As indicated by McDonald (Bertram 1969) integral calculation methods (solving ordinary differential equations for integral parameters) usually depend on the transformation of an incompressible-flow method, and stand or fall with the transformation. McDonald's comparisons appear to us to show that transformed integral methods generally give rather less plausible answers than his 'modified mixing length' and eddy-viscosity methods, although McDonald does not draw this conclusion and the unreliability of the experimental data makes any definite assessment impossible. One integral method which does not use a transformation is that of Green (1968): this relies on an empirical correlation of the variation of dimensionless entrainment rate with Mach number in constant-pressure boundary layers (see figure 1). Green points out that the transformations he considered all gave a dimensionless entrainment rate independent of Mach number, contrary to experiment. The incompressible calculation method on which Green's method is based is the original entrainment procedure of Head (1958) which is less accurate than some more modern methods (Kline *et al.* 1969) so that direct comparisons of Green's method with others would give a misleading impression of his empirical treatment of compressibility. Without necessarily implying that integral methods as such are less reliable, it appears that the reliability of this type of empirical treatment is better assessed by considering differential methods (solving partial differential equations for local velocity).

Recently, three methods of this type have been published, by Herring & Mellor (1968), Cebeci, Smith & Mosinskis (1970) and Sivasegaram (1970): all are extensions of well-established methods for incompressible flow and the assumptions made about the effects of compressibility on the turbulent shear stress are mathematically simple. All three methods rely on simple 'local equilibrium' relations between velocity gradient and shear stress, ignoring the effects of turbulence history. A fourth recent method is a compressible version

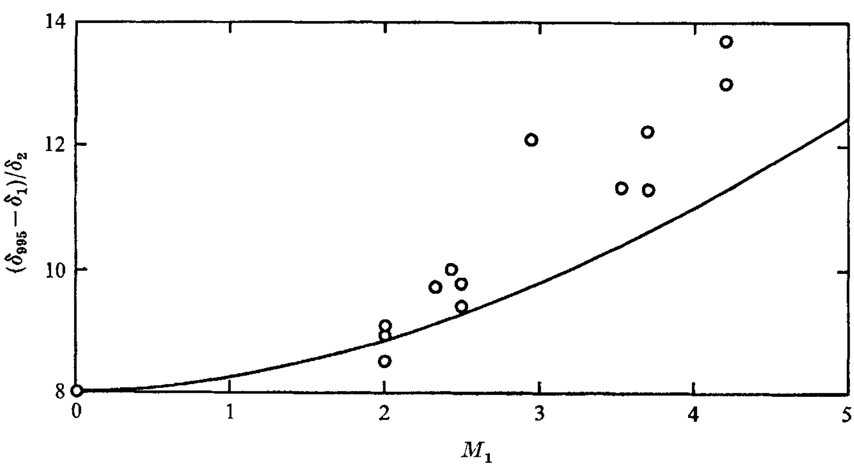

FIGURE 1. Approximate entrainment parameter $(\delta_{995}-\delta_1)/\delta_2$. O, data of various experiments; —, $8(\rho_1/\rho_{\frac{1}{4}})^{\frac{1}{2}}$, used in calculations.

of the eddy viscosity method of Ng & Spalding (1970). The eddy viscosity is taken to be the product of a turbulence velocity scale (the square root of the turbulent kinetic energy per unit mass) and a turbulence length scale: semi-empirical partial differential equations are solved for both these quantities and it is assumed that not only the empirical constants but also the terms in the equations are unaltered by compressibility, providing that the local mean density is inserted. The early report on this method, by Ng & Sivasegaram (1970), does not give any explanation of this assumption so that it cannot yet be assessed. Finally, the method of Donaldson & Rosenbaum (Bertram 1969), which employs differential equations for $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ as well as the shear stress $-\overline{uv}$, is evidently intended for extension to high-speed flow and could use Morkovin's hypothesis in the same way as the present method, but no details are available at present. Therefore, we can compare the present assumptions only with those used in the methods of Herring & Mellor, Cebeci *et al.* and Sivasegaram.

In the first two of these methods it is assumed that the apparent eddy viscosity in the outer part of the boundary layer is a constant multiple of

$$\int_0^\delta (U_1 - U)\,dy,$$

independent of the mean density profile and Mach number. Sivasegaram, in his extension to compressible flow of the mixing-length method of Spalding and Patankar (1967), assumes that (mixing length)/δ is independent of Mach number. The direct justification for these empirical assumptions is the analysis of data in compressible boundary layers in zero pressure gradient by Maise & McDonald (1968), Sivasegaram (1970) and L. C. Squire (Cambridge University, private communication). In view of the lack of data it is not possible to assess these empirical assumptions in compressible flow with pressure gradient. In incompressible flow large variations in dimensionless eddy viscosity or mixing length can occur in rapidly changing boundary layers (see, for example, Bradshaw & Ferriss 1965): rapidly changing boundary layers are of particular importance in high-speed flow and changes in density *per se* may affect the eddy viscosity or mixing length.

It will be shown in §3 that, as in incompressible flow, the mixing-length formula is a first approximation to the shear stress equation on which the present method is based: therefore, as in incompressible flow the present method is virtually certain to be an improvement on methods based on the mixing-length formula because it allows for 'history' effects on the turbulence. To show that large differences between it and the local-equilibrium methods can occur in practical cases, figure 8 compares the present method and the mixing-length method for a fictitious separating boundary layer. There are appreciable differences in predicted separation position, in the sense to be expected from the neglect of 'history' effects in the mixing-length expression. Similar discrepancies are likely to arise with other local-equilibrium methods.

The method described in this paper was completed, and written up as an 'unpublished' report, in 1966. We have delayed publication in the hope that reliable test data for supersonic boundary layers in pressure gradient might appear: Sivasegaram (1970) has described his measurements in a small supersonic tunnel which we built for this purpose but which was unfortunately restricted to small pressure gradients. There are still no really satisfactory test cases, but in view of the publication of the other methods mentioned in the last paragraph, with their physical shortcomings, we felt we should now publish our work. We have not presented comparisons with all the available test cases because of their unsatisfactory nature, as documented in §7 below: for the same reason extensive comparisons with other predictions of these test cases would add little to the above discussion of the physical basis of the different methods.

We are cautiously optimistic about the accuracy of the method in supersonic flow in arbitrary pressure gradient: at the least, it is a rational technique for extending the method of Q to compressible flow and should maintain the accuracy of that method up to transonic Mach numbers or higher.

2. Derivation of the equations for compressible flow

Morkovin (Favre 1964) and Favre (1965) have discussed this question from the physical and mathematical points of view respectively. Here, our chief concern is the making of legitimate approximations to simplify the equations. Where only small letters appear in an equation they represent instantaneous (mean plus fluctuating) quantities: where both large and small letters appear they represent mean and fluctuating velocities respectively; overbars and primes are used to denote the mean and fluctuating parts of p and ρ. Tensor notation, with the repeated-suffix summation convention, is used in this section only.

The mean continuity equation

Taking the time mean of

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i) = 0, \tag{2}$$

we have

$$\frac{\partial}{\partial x_i}(\bar{\rho} U_i+\overline{\rho' u_i}) = 0. \tag{3}$$

The mean momentum equation

Taking the mean of

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}+\text{complicated viscous terms} \tag{4}$$

(the equation is given in full by Howarth (1953, p. 50)) and using the continuity equation, we have

$$(\bar{\rho} U_j+\overline{\rho' u_j})\frac{\partial U_i}{\partial x_j}+\overline{\rho' u_i}\frac{\partial U_j}{\partial x_j}+U_j\frac{\partial \overline{\rho' u_i}}{\partial x_j}$$
$$= -\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_j}(\bar{\rho}\,\overline{u_i u_j}+\overline{\rho' u_i u_j}) \text{ plus viscous terms.} \tag{5}$$

The turbulent kinetic energy equation

Multiplying the instantaneous momentum equation by u_i, the instantaneous continuity equation by $\frac{1}{2}u_i^2$, adding and taking the mean, we have

$$\frac{\partial}{\partial x_j}\tfrac{1}{2}\overline{\rho u_i^2 u_j} = -\overline{u_i\frac{\partial p}{\partial x_i}}+\text{viscous terms,} \tag{6}$$

where the symbols still denote instantaneous quantities. Changing the suffix on the right-hand side from i to j and discarding the terms in the *mean* energy equation, obtained by multiplying by the *mean* instead of the instantaneous velocity, we get

$U_j\frac{\partial}{\partial x_j}(\frac{1}{2}\bar{\rho}\overline{u_i^2}+\frac{1}{2}\overline{\rho' u_i^2})$	advection
$+\overline{\rho' u_i}\,U_j\frac{\partial U_i}{\partial x_j}$	turbulent mass flux times mean acceleration
$+(\frac{1}{2}\bar{\rho}\overline{u_i^2}+\frac{1}{2}\overline{\rho' u_i^2})\frac{\partial U_j}{\partial x_j}$	normal stress times mean dilatation
$-\overline{p'\frac{\partial u_j}{\partial x_j}}$	pressure-dilatation mean product
$+(\bar{\rho}\overline{u_i u_j}+\overline{\rho' u_i u_j})\frac{\partial U_i}{\partial x_j}$	turbulence production
$+\frac{\partial}{\partial x_j}(\overline{p' u_j}+\frac{1}{2}\bar{\rho}\overline{u_i^2 u_j}+\frac{1}{2}\overline{\rho' u_i^2 u_j})$	energy diffusion
$+$ viscous terms	
$= 0.$	(7)

The form of the equation given here is the same as Favre's equation (95).

Approximations

The above results are exact: we now proceed to discard some of the terms, using order-of-magnitude arguments based on the boundary-layer approximation and the relative smallness of the terms involving the density fluctuation. We shall require $(\gamma-1)M_1^2$ to be not much larger than unity because this allows us to eliminate nearly all such terms. This condition is not quite the same in spirit as Morkovin's: in effect it is a (very conservative) condition that

$$\overline{\rho' u_1^2} \ll \bar{\rho}\overline{u_1^2},$$

whereas Morkovin's is a condition that $\bar{\rho}\overline{u_1^2} \ll \bar{p}$.

Before using these order-of-magnitude arguments we must show that, outside the viscous sublayer, the viscous terms in the turbulent energy equation may be equated to the dissipation. Tritton (1961) has pointed out that viscosity fluctuations, caused even by small temperature fluctuations, may greatly increase terms like

$$\overline{\frac{\partial}{\partial x_j}\left[(\bar{\mu}+\mu')\frac{\partial(U_i+u_i)}{\partial x_j}\right]} \equiv \frac{\partial}{\partial x_j}\left(\bar{\mu}\frac{\partial U_i}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(\overline{\mu'\frac{\partial u_i}{\partial x_j}}\right),$$

because $\partial u_i/\partial x_j$ is much greater than $\partial U_i/\partial x_j$. However, Tritton's argument is based on the assumption that

$$\overline{\frac{\partial T}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\Big/\left[\overline{\left(\frac{\partial T}{\partial x_j}\right)^2}\,\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}\right]^{\frac{1}{2}}$$

is of order unity for some of the components, and this is unlikely to be the case for flows at high Reynolds number where most of the contribution to $\partial u_i/\partial x_j$ comes from the locally isotropic eddies, because by definition the strong correlation between the temperature fluctuations and the longitudinal velocity fluctuations cannot extend into the isotropic range: we are grateful to Dr Tritton for pointing this out. Tritton's argument will certainly apply near the viscous sublayer, but it is probably justifiable to apply incompressible-flow arguments for the neglect of the viscous terms outside the viscous sublayer if the boundary-layer Reynolds number is high. The sublayer thickness appears to be given roughly by $(-\overline{uv})^{\frac{1}{2}}y/\nu_{\text{local}} \approx 30$, as in incompressible flow.

Morkovin, supported by the experiments of Kistler (1959), shows that the fluctuation of total temperature in the flow is much less than the fluctuation of the static temperature, so that the latter is given to fair accuracy in a boundary layer by

$$c_p\,dT + U\,du = 0 \tag{8}$$

(from here onward we use u, v, w for the u_i and $q^2 = u^2+v^2+w^2$ for u_1^2).

Remembering that the pressure fluctuations are small if the Mach number fluctuation is much less than unity we obtain

$$\rho'/\bar{\rho} \approx (\gamma-1)M^2 u/U, \tag{9}$$

where M is the local Mach number. Using this relation we see

(i) $$\overline{\rho' u}/\bar{\rho}U \approx (\gamma-1)\,M^2\overline{u^2}/U^2 \ll 1; \tag{10}$$

(ii) since $V/U \sim d\delta_1/dx \sim d\delta_2/dx \sim \tau_w/\rho_1 U_1^2$ in small pressure gradients,† then

$$\overline{\rho' v} \simeq (\gamma-1)\,M^2\bar{\rho}\overline{uv}/U \sim \bar{\rho}V. \tag{11}$$

Therefore in general we cannot neglect $\overline{\rho' v}$ with respect to $\bar{\rho}V$.

(iii) $$\overline{\rho' q^2}/\bar{\rho}\overline{q^2} \simeq (\gamma-1)\,M^2\overline{q^2u}/\overline{q^2}U \sim (\gamma-1)\,M^2\overline{q^2v}/\overline{q^2}U_1 \ll 1, \tag{12}$$

since $\overline{q^2v}/\overline{q^2}$ is less than or equal to the entrainment velocity V_p at the edge of the flow, where 'advection' = 'diffusion' (see § 4). Over most of the boundary layer $\overline{q^2v}/\overline{q^2} \ll V_q$ so the inequality is very well satisfied if $(\gamma-1)\,M_1^2V_p/U_1 \ll 1$, generally equivalent to $(\gamma-1)\,M_1^2\,d\delta/dx \ll 1$ (see table 1).

(iv) Similarly $\overline{\rho' uv}/\bar{\rho}\overline{uv} \ll 1$. It is arguable that $\bar{\rho}\overline{uv}+\overline{\rho' uv}$, rather than $\bar{\rho}\overline{uv}$, could be considered as the basic variable, rendering this inequality unnecessary.

(v) Reverting for convenience to the suffix notation, we see that $\overline{\rho' u_i}\,\partial U_j/\partial x_j$ and $U_j\,\partial\overline{\rho' u_i}/\partial x_j$ are at most of the same order as $\overline{\rho' u}\,\partial U/\partial x$ which is negligible compared with $\bar{\rho}U\,\partial U/\partial x$ on account of (i).

(vi) Finally, $\overline{p'\partial u_j/\partial x_j} = -\overline{(p'/\rho)(\partial\rho'/\partial t + U_j\,\partial\rho'/\partial x_j)}$ which is small compared to, say, $\partial(\overline{p' u_j})/\partial x_j$ because the second factor is the rate of change of ρ' following the mean motion of the fluid; this is much smaller than $U_j\partial\rho'/\partial x_j$ if Taylor's hypothesis is obeyed.

The argument that $\overline{\rho' u_i^2 u_j}/\bar{\rho}\overline{u_i^2 u_j} \ll 1$ rests on dubious estimates of the fourth-order correlations and on the smallness of $\overline{q^2}/U_1^2$ compared with $d\delta/dx$ and is not to be relied on: fortunately we can consider all three terms in the energy diffusion together.

The terms in the turbulent energy equation that represent advection and normal stress times mean dilation may be written together as

$$\partial(\bar{\rho}U_j\,\tfrac{1}{2}\overline{u_i^2})/\partial x_j$$

which on using the continuity equation becomes

$$(\bar{\rho}U_j+\overline{\rho' u_j})\,\partial(\tfrac{1}{2}\overline{u_i^2})/\partial x_j - \partial(\tfrac{1}{2}\overline{u_i^2\rho' u_j})/\partial x_j.$$

Both the second term in this expression and the term in the turbulent energy equation representing mean acceleration times turbulent mass flux (or, roughly, shear-stress gradient times turbulent mass flux) are small compared with a typical value of the turbulence production, by virtue of the boundary-layer approximation, but these terms may play an appreciable part in the local energy balance near the outer edge of the layer where the turbulent energy is small, although they go to zero at the edge faster than $\overline{q^2}$. While this is not likely to affect the overall behaviour of the layer very much it is possible to represent these terms together roughly by a term

$$D \equiv \frac{tr}{a_1}(\gamma-1)\frac{M^2}{U}\tau\frac{\partial\,\tau/\bar{\rho}}{\partial y},$$

† Here δ_1 and δ_2 are the displacement and momentum thicknesses respectively.

where t is a constant such that tr/a_1 is about 7, and where density and velocity gradients have been assumed small. The effect of including this term is to increase the skin friction coefficient, $c_f \equiv \tau_w/\frac{1}{2}\bar{\rho}_1 U_1^2$, in zero pressure gradient by about 0·8 % at $M = 2$ and 1·5 % at $M = 3$. Rotta (1967) has pointed out that the term $\overline{\rho' v} U \partial V/\partial x$ may be appreciable if the surface curvature is large: we return to the effects of surface curvature in § 6.

We can now write the simplified equations as follows, for the case of two-dimensional mean flow, noting that according to the boundary-layer approximation some derivatives in the x direction are negligible compared to those in the y direction:

$$\frac{\partial \bar{\rho} U}{\partial x} + \frac{\partial}{\partial y}(\bar{\rho} V + \overline{\rho' v}) = 0, \tag{13}$$

$$U\frac{\partial U}{\partial x}\left(V + \frac{\overline{\rho' v}}{\bar{\rho}}\right)\frac{\partial U}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial \tau}{\partial y}, \tag{14}$$

$$\bar{\rho} U \frac{\partial \frac{1}{2}\overline{q^2}}{\partial x} + \bar{\rho}\left(V + \frac{\overline{\rho' v}}{\bar{\rho}}\right)\frac{\partial \frac{1}{2}\overline{q^2}}{\partial y} = \tau\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}(\overline{p' v} + \tfrac{1}{2}\bar{\rho}\overline{q^2 v} + \tfrac{1}{2}\overline{\rho' q^2 v}) - D - \bar{\rho}\epsilon, \tag{15}$$

where ϵ is the rate of dissipation of turbulent kinetic energy per unit mass.

These equations are valid only if $(\gamma - 1) M_1^2 \overline{u^2}/U_1^2 \ll 1$ (a simpler and rather more restrictive condition, applicable in mild pressure gradients, is

$$(\gamma - 1) M_1^2 d\delta/dx \ll 1)$$

but it is interesting to observe that Morkovin's simplification *may* be valid at all Mach numbers. If we represent a typical fluctuating Mach number in a constant-pressure flow by

$$(\tau/\rho)^{\frac{1}{2}}/a = (\tau/\gamma p)^{\frac{1}{2}} \leqslant (\tau_w/\gamma p)^{\frac{1}{2}} \equiv M_1(\tfrac{1}{2}c_f)^{\frac{1}{2}},$$

we find that, since according to Spalding & Chi's (1964) or Coles's (1964) correlation $c_f \sim 1/M_1^2$ for $M > 7$ on an adiabatic wall, the fluctuating Mach number never exceeds 0·15 to 0·2. Since $\tau/\bar{\rho}\overline{q^2} \approx 0{\cdot}15$, the Mach number based on the r.m.s. velocity fluctuation reaches 0·6 to 0·8 so that moderate compressibility effects may occur. Unfortunately the effect of the mean density gradient on the large eddies increases monotonically with free-stream Mach number (see figure 1 and § 4), but there seems a good chance of extending the present type of calculation method well into the hypersonic range. Of course, most of the practical interest is in highly-cooled walls, for which c_f and the fluctuating Mach number are higher, so that even Morkovin's condition would break down at hypersonic Mach numbers.

3. Solution of the equations

It is convenient to use $\tau/\bar{\rho}$ rather than τ as a variable so that $(\tau/\bar{\rho})^{\frac{1}{2}}$ can be used as a local velocity scale for the turbulent motion. Also, $(V + \overline{\rho' v}/\bar{\rho})$ is clearly a more suitable variable than V alone. In the equations only, we will write τ for $\tau/\bar{\rho}$ and V for $(V + \overline{\rho' v}/\bar{\rho})$. We use the Crocco relation $c_p T + \frac{1}{2} r U^2 =$ constant to find the mean temperature (and hence the mean density and Mach number) in terms of U, and make the following definitions, with $\tau_{\max}$ taken as the maximum value in the *outer* layer, $y > \frac{1}{4}\delta$:

$$\left.\begin{aligned} a_1 &\equiv \tau/\overline{q^2} = \text{constant},\\ L &\equiv \tau^{\frac{3}{2}}/\epsilon,\\ G &\equiv \left(\frac{\overline{p'v}}{\bar{\rho}}+\tfrac{1}{2}\overline{q^2v}+\frac{1}{2}\frac{\overline{\rho' q^2 v}}{\bar{\rho}}\right)\Big/\tau\,\tau_{\max}^{\frac{1}{2}} \end{aligned}\right\} \tag{16}$$

as in incompressible flow. Equations (13) to (15) for the variables U, V, and τ become respectively

$$\frac{\partial U}{\partial x}(1+r(\gamma-1)M^2)+\frac{\partial U}{\partial y}\frac{V}{U}r(\gamma-1)M^2+\frac{\partial V}{\partial y}+\frac{U}{\bar{p}}\frac{d\bar{p}}{dx}=0, \tag{17}$$

$$U\frac{\partial U}{\partial x}+\left(V-r(\gamma-1)M^2\frac{\tau}{U}\right)\frac{\partial U}{\partial y}=\frac{-U^2}{\gamma M^2}\frac{1}{\bar{p}}\frac{d\bar{p}}{dx}+\frac{\partial \tau}{\partial y}, \tag{18}$$

$$\frac{U}{2a_1}\frac{\partial \tau}{\partial x}+\left(\frac{V}{2a_1}+\frac{tr}{a_1}(\gamma-1)M^2\frac{\tau}{U}\right)\frac{\partial \tau}{\partial y} = \tau\left(1+G\tau_{\max}^{\frac{1}{2}}r(\gamma-1)\frac{M^2}{U}\right)\frac{\partial U}{\partial y}-\tau_{\max}^{\frac{1}{2}}\frac{\partial}{\partial y}(G\tau)-\frac{\tau^{\frac{3}{2}}}{L}, \tag{19}$$

where the coefficient of dp/dx in the second (momentum) equation is really equal to $-1/\bar{\rho}$ as usual but is written in the above form so that the mean density does not appear explicitly in the equations. In order to derive the equations we need the Crocco relation for the temperature profile: we do *not* need any 'compressibility transformation' and none has been invoked in the derivation of the equations. Moreover, it is clear that no simple transformation will reduce these equations to their incompressible form. If the pressure gradient is zero, γ appears in equations (17)–(19) only in the group $r(\gamma-1)M^2$ so that the behaviour of boundary layers in other gases can be deduced from the behaviour in air at a different Mach number. This similarity is not found in arbitrary pressure gradients but as none of the empirical data depend on γ, with the possible exception of the additive constant in the logarithmic law (34), the method can be, and has been, applied to other gases.

The shear stress equation (19) reduces to the mixing-length formula, as in incompressible flow, if we neglect the 'transport' terms (i.e. those containing a_1 or G). Equation (19) becomes

$$\tau\,\partial U/\partial y = \tau^{\frac{3}{2}}/L, \tag{20}$$

which is the mixing-length formula with mixing length equal to L. The neglect of transport terms is justifiable by experiment near the surface but outside the viscous sublayer, and this is used below in dealing with the inner boundary condition. In the outer part of the flow, the mixing-length formula is a rough first approximation to (19) so that the present analysis can be used as a rather uncertain justification, via Morkovin's hypothesis, of the extension of 'mixing-length' methods to compressible flow.

Equations (17) to (19) are hyperbolic, like the incompressible-flow equations of Q, to which they reduce when $M \to 0$, and the method of characteristics is again used for solution. The characteristic angles are given by

$$\tan\gamma = \infty,\ \{V+(t-\tfrac{1}{2})W\tau+a_1GP \pm [\{a_1GP+(t+\tfrac{1}{2})\,W\tau\}^2+2a_1\tau(1+GPW)]^{\frac{1}{2}}\}/U, \tag{21}$$

where $$P = \tau^{\frac{1}{2}}_{\max}, \quad W = r(\gamma-1)\,M^2/U.$$

It might be thought that the vertical characteristic, along which the V component velocity propagates, should instead be inclined at the local Mach angle: however, the implicit use of the thin-boundary-layer approximation, $\partial p/\partial y = 0$, carries the assumption that pressure disturbances propagate vertically. The basic assumption is that the Mach angle is much greater than the angle of the inclined characteristics, implying $1/M_1 \gg d\delta/dx$: this is roughly equivalent to $(\gamma-1)\,M_1^2\,d\delta/dx \ll 1$ at non-hypersonic Mach numbers. Myring & Young (1968) discuss the question of normal pressure gradients in more detail: it becomes important on highly-concave surfaces where, however, the effects of curvature on the *turbulence* are extremely large (§6). The equations along the characteristics are

$$U\frac{dV}{dy}+(W\tau(1+UW)-V)\frac{dU}{dy}+(1+UW)\frac{d\tau}{dy}+\frac{U^2}{p}\frac{dp}{dx}\left(1-\frac{(1+UW)}{M^2\gamma}\right) = 0 \tag{22}$$

along the vertical characteristic, and

$$\begin{aligned}\tau(1+GPW)\frac{dU}{ds}-\frac{1}{2}\frac{d\tau}{ds}\left(GP+(t+\tfrac{1}{2})\frac{W\tau}{a_1}\pm\left[\left(GP+(t+\tfrac{1}{2})\frac{W\tau}{a_1}\right)^2+\frac{2\tau}{a_1}(1+GPW)\right]^{\frac{1}{2}}\right)\\ =\left[-\frac{\tau U}{M^2\gamma}\frac{1}{p}\frac{dp}{dx}(1+GPW)+\frac{\tau a_1}{U}\left(\frac{\tau^{\frac{1}{2}}}{L}+G'P\right)\left(GP+(t+\tfrac{1}{2})\frac{W\tau}{a_1}\right.\right.\\ \left.\left.\pm\left\{\left(GP+(t+\tfrac{1}{2})\frac{W\tau}{a_1}\right)^2+\frac{2\tau}{a_1}(1+GPW)\right\}^{\frac{1}{2}}\right)\right]\frac{dx}{ds}\end{aligned} \tag{23}$$

along the other two characteristics.

The inner 'boundary' condition, applied at one mesh length from the surface, is the universal inner layer law, the compressible version of the logarithmic law used at low speeds (see Q, p. 603). The general form for compressible flow of a given gas in small pressure gradient is (Rotta 1960)

$$U/u_\tau = f(u_\tau y/\nu_w, \quad u_\tau/a_w), \tag{24}$$

where $u_\tau \equiv (\tau_w/\rho_w)^{\frac{1}{2}}$, and $u_\tau/a_w \equiv M_\tau$ is the only relevant Mach number: M_1 does *not* appear. The numerical procedure for solving the inner boundary condition simultaneously with the equation on the ingoing inclined characteristic is similar to that used in incompressible flow. As in incompressible flow, more complicated cases such as surface roughness or transpiration can be accommodated if the corresponding inner law is known (Squire 1969). Near the surface, but outside the viscous sublayer, (19) reduces to the mixing-length formula (with mixing length $l = Ky$ where $K \approx 0{\cdot}40$) and the inner law for the more complicated cases can be obtained by integration, with a 'constant' of integration representing the effect of the sublayer. The 'constant' is actually a function of parameters like M_τ, dimensionless shear-stress gradient, roughness Reynolds number, ratio of transpiration velocity to u_τ, and so on. The constant can be found only from experimental velocity profiles or from a model of the turbulence in the viscous sublayer itself. The most sophisticated assumption made about the viscous sublayer to date is that l/Ky is a universal function of a local turbulence Reynolds number ($(-\overline{uv})^{\frac{1}{2}}y/\nu_{\mathrm{local}}$ being the most realistic definition). Unfortunately this overlooks the fact that significant energy diffusion towards the surface occurs so that 'local equilibrium' assumptions are not valid in the viscous sublayer. The local-equilibrium assumptions are found to give fairly good results in cases where profile measurements are available, but in these cases they are not really needed. In any case, the simplest way of incorporating the effect of the sublayer in a calculation method, however that effect is derived in the first place, is via the 'constant' of integration. Integration across the sublayer at each step of the calculation is time-consuming and unnecessary, because even if a turbulence model for the sublayer is being used the integrations can be done once for all to establish the 'constant' of integration as a function of the above-mentioned parameters. Sublayer integration at each step can be justified only by numerical convenience: the method of characteristics is better adapted than rectangular-mesh methods for matching to a complicated boundary condition because gradients need not be matched (see Ferriss 1969), so we have not been forced to integrate in this way.

The computer program is so similar in operation to the incompressible-flow program, described in Q and in greater detail by Ferriss & Bradshaw (1968), that no detailed explanation is needed: the major change is the introduction of the variable $W = r(\gamma-1)M^2/U$, which it is convenient to store and transfer in the same way as U. The program will not run at $M = 0$ exactly because division by M occurs, but the results of runs at very low Mach number compare satisfactorily with those of the incompressible program. The predictions of momentum thickness agree with the momentum integral equation to within one per cent of the change in δ_2, in mild pressure gradients. The running time of the program is no more than 20 % greater than that of the incompressible program. To make the program simpler to use in cases where the initial conditions are ill-defined or unimportant (preferably the latter as well as the former) we have inserted a routine for generating an initial velocity profile from Coles's family, given c_f and δ_2: a shear-stress profile is then generated using an assumed mixing-length distribution chosen to give the right shear-stress profile in zero pressure gradient in incompressible flow. Maise & McDonald's (1968) work suggests that the mixing-length assumption should be adequate up to $M = 5$: Coles's profile family is not very accurate above the transonic range, but should again be adequate for ill-defined initial conditions.

Like the program of Q, the present program contains an allowance in the continuity equation for lateral divergence or convergence.

4. Empirical data input and limits of validity

High Reynolds numbers

In the test cases presented here we have used exactly the same numerical values for the empirical functions a_1, L and G as were used in Q for incompressible flow. One ought not to expect the effect of mean density variations on turbulent energy diffusion to be entirely represented by inserting the *local* density in the definition of the energy diffusion function G (equation (16)) because the large eddies which diffuse energy extend across most of the outer layer: however, as will now be shown, the variation of diffusion with Mach number is adequately represented up to $M_1 = 3$ by this means, so we have not bothered to introduce an explicit function of M_1 into the specification of G. Near the edge of the boundary layer, diffusion from below is entirely responsible for the increase of turbulent energy along a streamline, and the turbulent energy equation can be reduced to

$$\bar{\rho} V_p \frac{\partial \frac{1}{2}\overline{q^2}}{\partial y} = \frac{\partial}{\partial y}(\overline{p'v} + \tfrac{1}{2}\bar{\rho}\overline{q^2 v} + \tfrac{1}{2}\overline{\rho' q^2 v}), \tag{25}$$

neglecting the D term (§2) which appears to go to zero at the edge of the layer more rapidly than the retained terms. Assuming that the product of $\bar{\rho}$ and the entrainment velocity V_p can be regarded as independent of y near $y = \delta$, integration gives

$$V_p = \left(2\frac{\overline{p'v}}{\bar{\rho}} + \overline{q^2 v} + \frac{\overline{\rho' q^2 v}}{\bar{\rho}}\right)\Big/\overline{q^2} \tag{26}$$

$$\approx \text{constant near } y = \delta,$$

(so that since $\overline{q^2 v}/\overline{q^2}$ reaches a maximum near $y = \delta$ it is elsewhere less than V_p). Substituting from (16), we have

$$[(\tau/\bar{\rho})_{\max}]^{\frac{1}{2}}(2a_1 G)_{y=\delta} = V_p \tag{27}$$

and V_p is *defined* by

$$\bar{\rho}_1 V_p = \frac{d}{dx}\bar{\rho}_1 U_1(\delta - \delta_1), \tag{28}$$

where we can take δ as δ_{995}, the value of y at which $U = 0{\cdot}995 U_1$. This relates the value of G at $y = \delta$ to the entrainment rate, which is easily deduced from mean velocity profiles. In zero pressure gradient, where $(\tau/\rho)_{\max}$ as defined in §2 occurs *at* $y = \frac{1}{4}\delta$,

$$\left(\frac{\bar{\rho}_1}{\bar{\rho}_{\frac{1}{4}}}\right)^{\frac{1}{2}}(a_1 G)_{y=\delta} \propto V_p\left(\frac{\rho_1}{\tau_w}\right)^{\frac{1}{2}}, \tag{29}$$

where $\bar{\rho}_{\frac{1}{4}}$ is the mean density at $y = \frac{1}{4}\delta$ and the constant of proportionality is independent of Mach number if $\tau_w/\tau_{\frac{1}{4}}$ is constant, which appears to be a good approximation. In zero pressure gradient, where

$$d(\delta - \delta_1)/dx \quad \text{is nearly} \quad [(\delta - \delta_1)/\delta_2]\, d\delta_2/dx \quad \text{and} \quad d\delta_2/dx = \tau_w/\rho_1 U_1^2,$$

we find on substituting the definition of V_p that

$$\left(\frac{\rho_1}{\rho_{\frac{1}{4}}}\right)^{\frac{1}{2}}(a_1 G)_{y=\delta} \propto \frac{\delta - \delta_1}{\delta_2}\left(\frac{\tau_w}{\rho_1 U_1^2}\right)^{\frac{1}{2}}. \tag{30}$$

In incompressible flow, experimental data for entrainment are well represented by taking

$$G = \left(\frac{\tau_{\max}}{\rho_1 U_1^2}\right)^{0\cdot 50} f\left(\frac{y}{\delta_{995}}\right), \tag{31}$$

and if the same function is assumed in compressible flow we finally obtain, as a result of these assumptions,

$$\frac{\delta - \delta_1}{\delta_2} \propto \left(\frac{\rho_1}{\rho_{\frac{1}{4}}}\right)^{\frac{1}{2}}. \tag{32}$$

This supplies roughly the required variation with Mach number (figure 1). The arguments for using $(\tau/\bar{\rho})_{\max}$ rather than $(\tau_{\max}/\rho_1)$ are not strong: the latter would overestimate the effect of Mach number on entrainment by about the same amount that it is now underestimated, and would agree better with the more rapid variation of entrainment with Mach number suggested by Green (1968). However, the calculations are not very sensitive to the choice of G; as in low-speed flow, L, the 'generalized mixing length', is much the most important of the three empirical functions. The final representation of energy diffusion is

$$\left(\frac{\overline{p'v}}{\bar{\rho}} + \tfrac{1}{2}\overline{q^2 v} + \frac{1}{2}\frac{\overline{\rho' q^2 v}}{\bar{\rho}}\right) \equiv G\frac{\tau}{\rho}\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}}_{\max}. \tag{33}$$

The other empirical data needed are (i) the recovery factor r, taken as 0·89 for air; (ii) the viscosity temperature exponent, taken as 0·76 for air; (iii) the ratio of specific heats, γ, taken as 1·4 for air; and (iv) the universal inner layer profile. Inspection of published data showed that for $M < 3$ and $y/\delta < 0{\cdot}05$ the relation between U and τ_w in zero pressure gradient is given adequately by the incompressible logarithmic law if fluid properties are evaluated at the wall, giving

$$\frac{U}{u_\tau} = \frac{1}{K}\left(\log_e \frac{u_\tau y}{\nu_w} + A\right), \tag{34}$$

and so we have used $K = 0{\cdot}4$, $A = 2$ as in incompressible flow. As in the case of G, an empirical Mach number dependence, or the more complicated analytical expression of Rotta (1960) could be inserted without difficulty. Calculations in zero pressure gradient show that the apparent K (i.e. the reciprocal of the slope of the logarithmic part of the profile in U/u_τ, $u_\tau y/\nu$ co-ordinates) rises to about 0·45 at $M_\tau = 0{\cdot}05$ ($M_1 \approx 2$) and 0·65 at $M_\tau = 0{\cdot}1$ ($M_1 \approx 5$), but the actual departure from (34) is small. As in incompressible flow, we use Townsend's (1961) modification to the logarithmic law when the shear-stress gradient in the y direction is large: an analytic function of $(y/\tau_w)\,\partial\tau_w/\partial y$, given in Q, is added to the right-hand side of (34). Our attempt to include compressibility effects in Townsend's modification led to very complicated, and uncertain, expressions and so we have used the incompressible form, which differs little from the more complicated expression except near separation at high supersonic Mach numbers.

Low Reynolds numbers

A final piece of information, required both in compressible and incompressible flow, is the effect of Reynolds number on a_1, L and G. It is well established that at high Reynolds number the energy-containing part of the turbulence does not depend directly on viscosity: this leads to Reynolds-number independence of a_1, L and G and to such well-known formulae as the 'defect law'

$$\frac{U_1 - U}{(\tau_w/\rho_w)^{\frac{1}{2}}} = f\left(\frac{y}{\delta}\right) \tag{35}$$

for the outer part of an incompressible boundary layer in zero pressure gradient. However, Coles (1962) found that the defect law changed with Reynolds number for $U_1\delta_2/\nu < 5000$, or

$$(U_1\delta/\nu)\,(\tfrac{1}{2}c_f)^{\frac{1}{2}} \equiv (\tau_w/\rho_w)^{\frac{1}{2}}\,\delta/\nu < 2000$$

(the latter Reynolds number being the more useful for comparing different flows if δ is understood as the thickness of any shear layer). This trend has been roughly simulated in the incompressible version of the present method by allowing the dissipation length parameter L to vary as $[(\tau_{max}/\rho_1)^{\frac{1}{2}}\,\delta/2000\,\nu_1]^{-\frac{1}{8}}$ in the outer layer only (Simpson 1970) as long as this quantity is greater than unity. Of course a_1 and G probably vary as well, but L is the most important parameter. There has been no comparably thorough analysis of compressible data but the measurements of Hastings & Sawyer (1970) at $M_\infty = 4$ show that the same effect occurs. Herring & Mellor (1968) found that comparison between their calculation method and experiment was optimized by multiplying the eddy viscosity in the outer layer by

$$1 + (1100\,\nu_1/U_1\delta_1)^2$$

roughly equivalent to $1 + [400\,\nu_1/(\tau_{max}/\rho_1)^{\frac{1}{2}}\,\delta]^2$. Even allowing for the fact that a given percentage change in eddy viscosity produces about twice the effect of the same percentage change in dissipation length parameter, this is a rather larger adjustment than that of L mentioned above, but it probably reflects differences between the two calculation methods more than differences between compressible and incompressible flow. The effect is very much larger than the direct effect of viscous shear stress. More work is needed before low-Reynolds-number effects can be properly represented in any calculation method: in compressible flow, the effect of viscosity fluctuations on the dissipation may be important. The best course for users of the present method is to do calculations with the original distribution of L (given in Q) and with a distribution factored by $[(\tau_{max}/\rho_1)^{\frac{1}{2}}\,\delta/2000\,\nu_1]^{-\frac{1}{8}}$ in the outer layer, retaining $L = 0{\cdot}4y$ in the inner layer: the difference between the two calculations gives an idea of the effects of low Reynolds numbers and the latter calculation may be relied on if these effects are not too large.

It remains to discuss the minimum Reynolds number below which the method *cannot* be used. A plausible criterion for reverse transition from turbulent to laminar flow, and thus a lower limit of validity of any turbulent calculation method, was given by Bradshaw (1969*a*) as $[(\tau/\rho)^{\frac{1}{2}}\,L/\nu]_{max} \approx 12$, for incompressible flow. In mild pressure gradients the maximum value of this quantity is reached at about $y = 0{\cdot}2\delta$. If we extend the criterion to compressible flow at constant pressure simply by substituting local values of density and viscosity according to the Crocco formula for an adiabatic wall, which is plausible because the original criterion was based on local-equilibrium arguments, we obtain

$$(U_1\delta/\nu_1)\,(\tfrac{1}{2}c_f)^{\frac{1}{2}} \approx 150\,(1+0{\cdot}07\,M_1^2)^{1{\cdot}26} \tag{36}$$

as the lowest Reynolds number at which turbulent flow can be maintained (we have applied the criterion at $y = 0{\cdot}2\delta$ and assumed $U/U_1 = 0{\cdot}8$ there). Using typical values for c_f and δ_2/δ we find that the minimum permissible value of $U_1\delta/\nu_1$ rises from 3000 at $M_1 = 0$ to 18000 at $M_1 = 5$: $U_1\delta_2/\nu_1$ increases from about 300 at $M = 0$ to 750 at $M = 5$. It is not possible to compare these values with the Reynolds number at which transition from *laminar* to *turbulent* flow occurs because the latter depends on the disturbances that cause transition. In strongly accelerated flows, reverse transition occurs at higher values of $U_1\delta/\nu$ because τ decreases rapidly with distance from the wall: in the experiments of Michel, Quemard & Elena (1969) at $M \approx 2$, departures from the logarithmic law similar to those occurring in reverse transition at low speeds were found at $U_1\delta/\nu_1 \approx 70000$. In general, some care is needed in applying calculation methods at low Reynolds numbers.

5. Test cases in zero pressure gradient

The results for skin-friction coefficient c_f in zero pressure gradient at Mach numbers from 0 to 5 are shown in table 1. The Reynolds numbers were chosen so that the output velocity profile from one run could be used to start the next: the accuracy of the calculations does *not* depend on Reynolds number in this

		c_f			H_i		
M_1	$\dfrac{U_1\delta_2}{\nu}$	Calculated	Spalding–Chi	Winter–Gaudet	Calculated	Winter–Gaudet	$(\gamma-1)M_1^2\dfrac{d\delta}{dx}$
0	12400	0·00246	0·00252	0·00256	1·33	1·31	0
1	15200	0·00217	0·00222	0·00222	1·32	1·30	0·004
2·2	50000	0·00139	0·00142	0·00138	1·27	1·26	0·016
3	71000	0·00105	0·00104	0·00107	1·25	1·24	0·028
4	149000	0·000714	0·000728	0·000752	1·20	1·22	0·042
5	50000	0·000711	0·000702				0·083

TABLE 1. Results of calculations in zero pressure gradient: adiabatic wall, recovery factor = 0·89.

range. Since several correlations of experimental data exist we have compared with these rather than with any restricted set of experiments. Agreement with the correlation of Spalding & Chi (1964) is good and generally within the accuracy of using their tables. The results also agree well with the recent correlation of Winter & Gaudet (1968) who found that the 'incompressible' shape factor H^i, (equal to δ_1^i/δ_2^i where δ_1^i and δ_2^i are the displacement and momentum thicknesses evaluated as in incompressible flow, excluding density variation) was a unique function of $U_1\delta_2^i/\nu_1$, independent of Mach number up to $M = 2{\cdot}8$ at least. Winter & Gaudet deduced a skin-friction law, similar in concept and numerical results to that of Spalding & Chi, and again the results of the present method are in good agreement. These results are satisfactory as far as they go, although it must be remembered that skin-friction predictions for zero pressure gradient are greatly influenced by the accuracy of the inner 'boundary' condition, applied where the velocity is already as high as $\frac{2}{3}U_1$: the assumptions made in the main part of the flow are therefore tested only one-third as severely as at

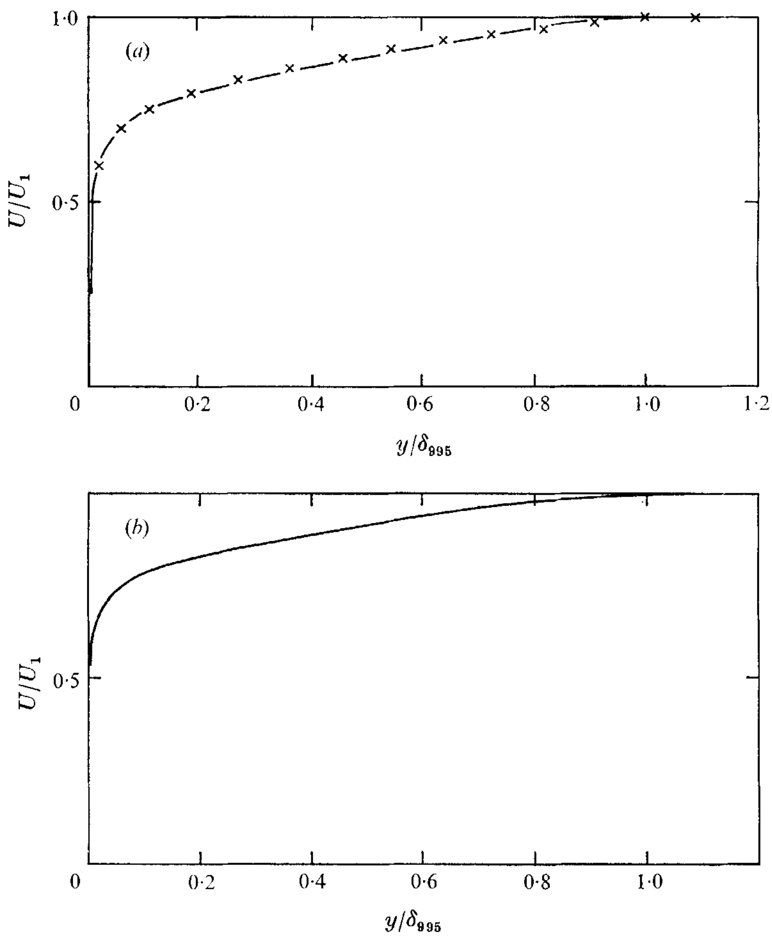

FIGURE 2. Calculated velocity profiles. (a) $M = 2{\cdot}2$, $U_1\delta_2/\nu = 50{,}000$. ×, data of Winter & Gaudet (1968); —, calculation. (b) $M = 4{\cdot}0$, $U_1\delta_2/\nu = 149\,000$.

first appears. However, the present method predicts the outer-layer velocity profiles (figure 2) quite accurately, and this agreement, together with the good agreement in c_f up to a Mach number where c_f is barely a third of its low-speed value, gives one fair confidence in the truth of Morkovin's hypothesis and the reliability of the present method.

A less direct but more thorough confirmation of Morkovin's hypothesis is given by the observation, mentioned in § 1, that the distribution of the dimensionless apparent mixing length, $l/\delta = (\tau/\rho)^{\frac{1}{2}}/(\delta\, \partial U/\partial y)$, in boundary layers in zero pressure gradient is almost independent of Mach number up to $M = 5$. Now as was shown in Q, l/L is equal to the ratio of turbulent energy dissipation to turbulent energy production: in zero pressure gradient this ratio is nearly unity everywhere and it is most unlikely that the ratio would change appreciably with Mach number. This implies that L/δ is almost independent of Mach number, as indicated by Morkovin's hypothesis and as assumed in the present calculations.

Maise & McDonald (1968) found that the dimensionless eddy viscosity, $\nu_\tau/U_1\delta_1 = \tau/(\rho U_1 \delta_1 \partial U/\partial y)$ varied strongly with Mach number. Herring & Mellor (1968) and Cebeci *et al.* (1969) have used δ_1^i, the 'incompressible' or 'kinematic' displacement thickness, as a length scale: $\nu_\tau/U_1\delta_1^i$ seems to be almost independent of Mach number. δ_1 and δ_1^i are both integral length scales representative of the mean velocity distribution and are in no way connected with typical eddy sizes, so that their use in predicting the turbulent shear stress can be justified only *a posteriori*: the total boundary-layer thickness δ, on the other hand, is equally the length scale of the largest eddies, which fill the boundary layer, and is therefore a plausible length scale for the turbulent motion as well as the mean motion.

Before making comparisons with experiments in pressure gradients we discuss one of the reasons why so many existing experiments are of doubtful value as test cases.

6. Effects of surface curvature

The most common way of generating a supersonic boundary layer in a pressure gradient is to use a curved surface, but if the pressure gradient is significant the effects of streamline curvature on the turbulence is likely to be significant also. The first-order analogy between buoyancy and streamline curvature used by Bradshaw (1969*b*) to estimate this latter effect, indicates that the dissipation length parameter or apparent mixing length will be multiplied by a factor

$$1\Big/\left[1+\beta\left(1+\frac{\gamma-1}{2}M^2\right)\left(2\frac{L}{R}\frac{U}{(\tau/\rho)^{\frac{1}{2}}}\right)\right],$$

where $\beta \approx 7$ on a convex surface (radius of curvature R greater than zero) and $\beta \approx 4$ on a concave surface. Now the relation between self-induced pressure gradient and surface curvature in an otherwise wave-free supersonic stream is

$$\frac{1}{\rho_1 U_1^2}\frac{dp}{dx} = \frac{-1}{R(M_1^2-1)^{\frac{1}{2}}} \tag{37}$$

and a convenient dimensionless pressure gradient, in compressible flow, is $(\delta_1/\tau_w)\,dp/dx$ which is the ratio of the two terms on the right-hand side of the momentum integral equation,

$$d(\rho_1 U_1^2 \delta_2)/dx = \tau_w + \delta_1\,dp/dx. \tag{38}$$

Substituting for R in terms of dp/dx in the mixing-length correction factor above, and taking $(2/c_f)^{\frac{1}{2}}$ as a rough value of $U/(\tau/\rho)^{\frac{1}{2}}$ the factor becomes

$$1\Big/\left[1-2\beta\frac{L}{\delta}\left\{\left(1+\frac{\gamma-1}{2}M_1^2\right)(M_1^2-1)^{\frac{1}{2}}(c_f/2)^{\frac{1}{2}}\frac{\delta}{\delta_1}\right\}\frac{\delta_1}{\tau_w}\frac{dp}{dx}\right],$$

where the factor in parentheses increases somewhat with Mach number. Taking for example $L/\delta = 0{\cdot}1$, $M = 2{\cdot}2$, $c_f = 0{\cdot}0016$, $\delta/\delta_1 = 5$, the mixing-length factor becomes

$$1\bigg/\left[1-0{\cdot}1\beta\frac{\delta_1}{\tau_w}\frac{dp}{dx}\right],$$

so if $\beta = 4$ (concave surface) very large changes in apparent mixing length are predicted when $(\delta_1/\tau_w)\,dp/dx$ approaches the modest value of unity. Since the mixing-length correction factor comes from an analogy which cannot reasonably be expected to hold to better than first order, we conclude that if self-induced pressure gradients on a curved surface are large, reliable predictions of boundary-layer development cannot be made at present. Whatever the detailed accuracy of the buoyancy-curvature analogy its order-of-magnitude plausibility seems to be adequate to support this conclusion. Further progress can be made only by direct study of the effects of streamline curvature on the turbulence structure, and since the simple analogy predicts that these effects depend strongly on Mach number (via the semi-empirical factor $1+\frac{1}{2}(\gamma-1)M^2$: see also Rotta 1967) it appears that this study should be extended to supersonic flow.

A related question is the behaviour of turbulence passing through oblique shock waves or expansions. This depends as much on the effects of dilatation as on the effects of streamline curvature. Neither our assumptions nor any other current assumptions could be expected to apply to situations like these.

It should be emphasized that the effects of curvature on the turbulence are not directly connected with the effects of curvature on the mean equations and in general they will be much more important than the latter, at least at the lower Mach numbers.

It should also be emphasized that most supersonic aircraft are sufficiently slender for curvature effects *and* self-induced pressure gradients to be small: large pressure gradients are found mainly in engines and intakes or in regions of interference between components, and in these cases the pressure gradient is generally imposed externally rather than by large surface curvature. In subsonic flow there is of course no direct connexion between surface curvature and pressure gradient, and the effects of curvature on the turbulence are usually small enough to be represented by a first-order correction.

7. Test cases in pressure gradient

Rejection of measurements on highly curved surfaces leaves very few suitable test cases. Before abandoning flows on curved surfaces we did calculations for the boundary layers of Clutter & Kaups (1964), McLafferty & Barber (1962) and Winter, Rotta & Smith (1968), the last-named being strongly affected by curvature on the waisted body used, even at the lowest Mach number, 0·6. In all cases the predictions without a curvature correction were poor. The predictions of McLafferty & Barber's flow were grossly changed by using the buoyancy analogy, which was pushed well beyond its likely limits of validity. The predictions for the $M = 2$ test case of Winter *et al.* (figure 3) were improved by an allowance for curvature but diverged from the experimental trend near the rear of the body (it is difficult to reconcile the high experimental values of c_f in this region with the adverse pressure gradient). Other workers (e.g. Herring & Mellor 1968) have found similarly poor agreement with the waisted-body experiment, and it seems certain that the development of the boundary layer was strongly affected, not only by longitudinal curvature but also by cross-sectional curvature and rapid increases of body radius. Note that the 'experimental' points for c_f in figure 3 are obtained from the measured velocity profiles using the logarithmic law: the measurement using surface pitot tubes seems to be too low in the region of the waist, and there is no valid reason to suspect the logarithmic law in this region.

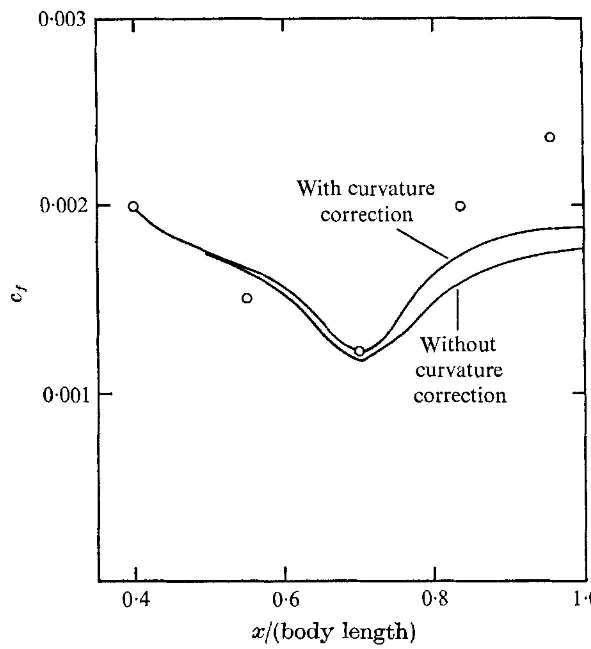

FIGURE 3. Waisted body of revolution (Winter *et al.* 1968). ○, experiment; —, calculation.

Comparisons with measurements at high subsonic speeds by Firmin & Cook (1968) show large differences between the predicted skin friction and that measured by surface pitot tubes: the predictions agree, within the likely experimental accuracy, with skin-friction values obtained using the logarithmic law (predictions by the present method published by Firmin & Cook were based on inconsistent input data, but the effect on c_f was small). These test cases, in which the local Mach number did not exceed unity, do not add significantly to the extensive comparisons for incompressible flow reported in Kline *et al.* (1969): at $M = 1$, skin-friction coefficients are only 6 or 7 % lower than at $M = 0$, so that very crude estimates of compressibility effects would serve to predict c_f to within the likely experimental error.

The three supersonic test cases with appreciable pressure gradients on flat surfaces are those of Pasiuk, Hastings & Chatham (1964), Zwarts (N.R.C., Canada, unpublished: we are indebted to Dr Zwarts for access to his data) and

Sivasegaram (1970). All leave something to be desired as test cases, and their various shortcomings will be discussed individually.

Pasiuk *et al.* did not measure skin friction directly, and values derived from their velocity profiles using the logarithmic law (figure 4) are almost exactly

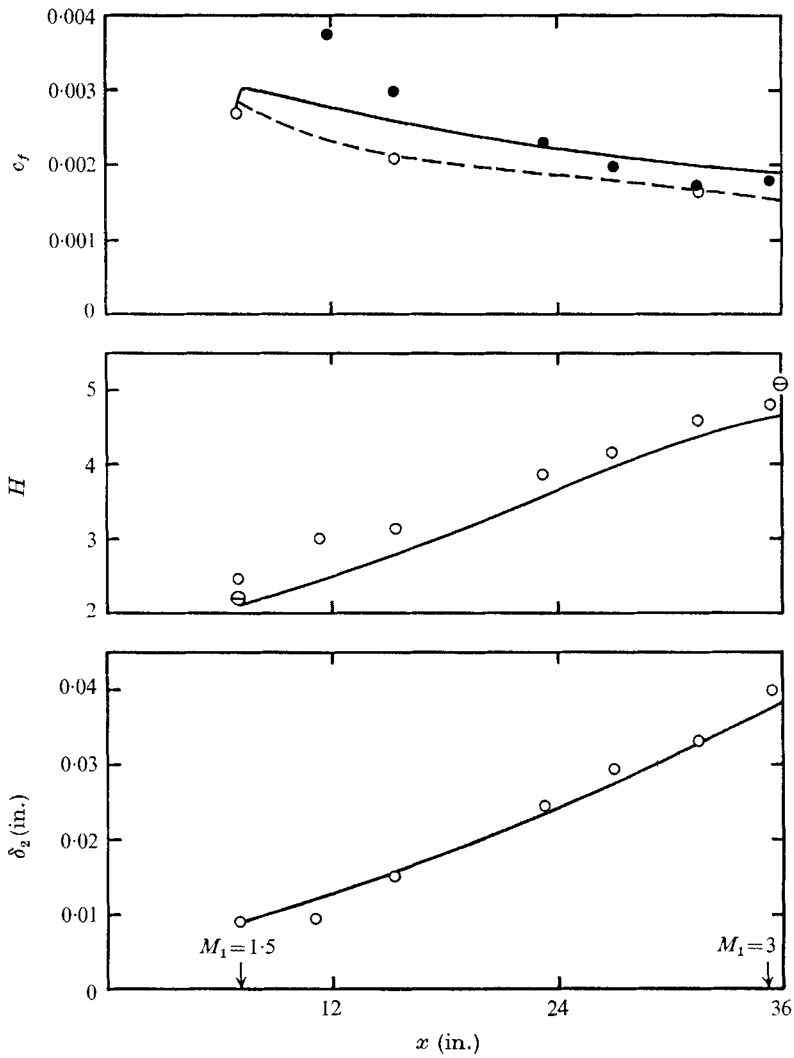

FIGURE 4. Accelerating flow of Pasiuk *et al.* (1964, zero heat transfer). ○, experiment (c_f from log law); —, calculation; ●, c_f from momentum balance, according to Cebeci *et al.* (1970); – – – –, c_f from 'flat-plate' formula of Spalding & Chi (1964); ⊖, calculation, with recovery factor taken as 1·0 for evaluating H.

the same as the skin friction in a *constant-pressure* boundary layer at the same Mach number and Reynolds number, despite the fact that $(\delta_1/\tau_w)\,dp/dx$ was of the order of -1, leading one to expect significant increases in c_f (as predicted by the present method). Values of c_f inferred from the momentum integral equation by Cebeci *et al.* are also shown in figure 4, but are not very reliable because of the difficulty of differentiating a curve fitted to experimental points and because of the possibility of lateral convergence or divergence of the flow.

Zwarts measured skin friction using surface pitot tubes: the values shown in figure 5 agree roughly with values obtained from logarithmic profiles. The

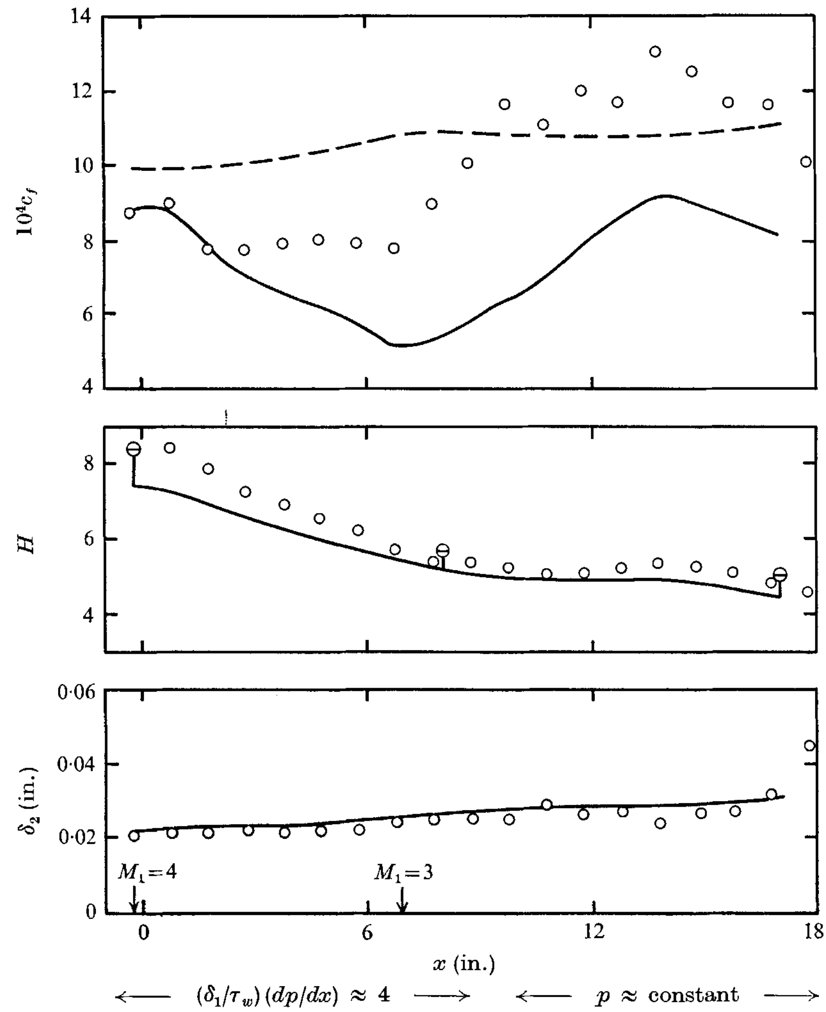

FIGURE 5. 'Relaxing' flow of Zwarts (retardation followed by constant-pressure region). ○, experiment (r = 1·0 for evaluating H); —, calculation with allowance for lateral convergence; — —, c_f from Spalding–Chi formula; ⊖, calculation, with r = 1·0 for evaluating H.

external Mach number decreased from 4·0 at $x = -0{\cdot}25$ in. to 3·0 at about $x = 15$ in. after which it decreased again very slightly: it is therefore surprising that experimental values of c_f in the latter half of the tunnel are significantly *larger* than in a constant-pressure boundary layer. The calculated skin friction coefficient approaches the constant-pressure value from below, as in the analogous low-speed boundary layer (run 2400 of Kline *et al.* 1969). The main shortcoming of Zwarts's experiment was the rather small tunnel width of 5 in. (roughly 12 boundary-layer thickness) compared with a working section length of 17 in. plus about the same distance to generate the initial boundary layer. Typically, the virtual origin deduced from the imbalance in the experimental momentum integral equation was about 25 in. downstream of the point considered although *divergence* occurred in the latter part of the constant-pressure

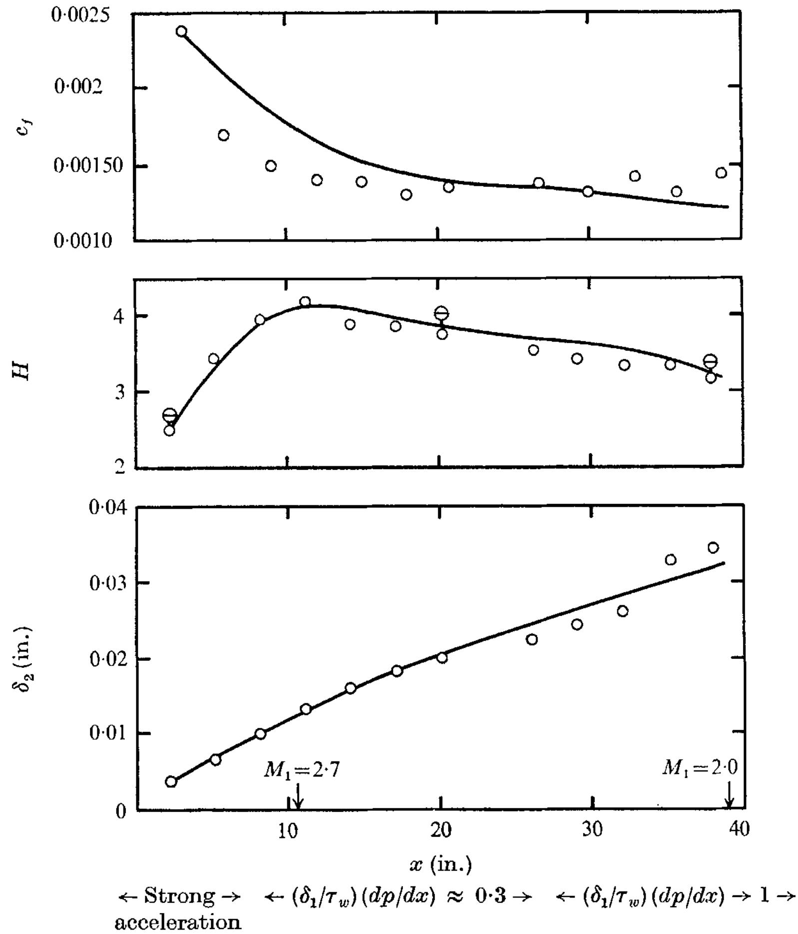

FIGURE 6. Retarded flow of Sivasegaram (1970).
○, experiment; —, calculation; ⊖, calculation, with $r = 1\cdot0$ for evaluating H.

region. In the calculations, the equivalent lateral convergence was inserted into the continuity equation, but this simple kinematic correction is unreliable if the convergence is large: the crossflow angle may vary through the boundary layer and the convergence as such, or the effect of the nearby sidewalls, may affect the turbulence. However, a kinematic correction is better than ignoring convergence altogether. The apparent convergence deduced from the momentum integral equation is much too large to be attributed to errors in c_f measurement.

Sivasegaram measured skin friction using small surface pitot 'fences' immersed in the viscous sublayer and calibrated in zero pressure gradient. The tunnel width was 12 in. or about 20 boundary-layer thicknesses and the momentum integral equation balanced to within the experimental accuracy. Unfortunately the pressure gradients obtained were rather mild $(-0{\cdot}5 < (\delta_1/\tau_w)\,dp/dx < 0{\cdot}5$ over most of the flow) and even the mixing-length formula, without allowance for turbulence history, will predict the result quite adequately. The measured velocity profiles suggest that the flow at the first few measurement stations was

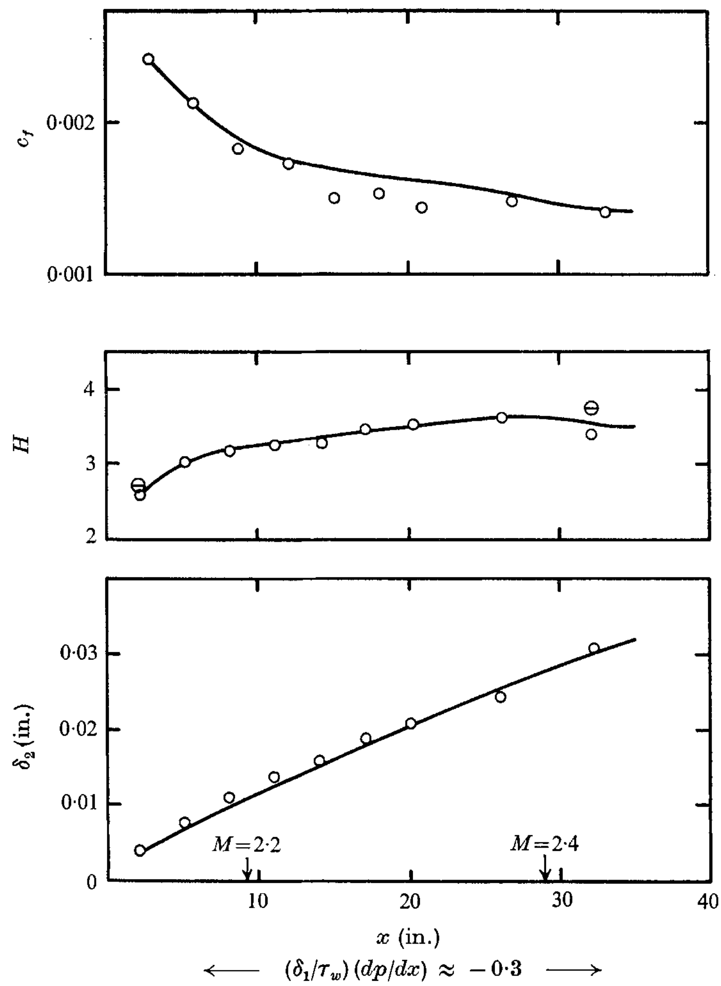

FIGURE 7. Accelerating flow of Sivasegaram (1970).
○, experiment; —, calculation; ⊖, calculation, with $r = 1{\cdot}0$ for evaluating H.

near a state of reverse transition because of the rapid acceleration through the nozzle, which would explain the disagreement between experimental and calculated c_f in figure 6. The acceleration in the nozzle region of the flow shown in figure 7 was much less severe.

It is *possible* that the measured c_f values in figures 4 and 5 are correct and representative of two-dimensional flow: the conclusion would be that in supersonic flow a favourable pressure gradient does not appreciably increase skin friction, and relaxation from an adverse pressure gradient causes a large overshoot in skin friction. Both these effects could occur if the lifetime of turbulent eddies were far longer in supersonic flow than in low-speed flow but none of our current physical ideas about turbulence support this. The value of H obtained for a given velocity profile depends strongly on the temperature profile (i.e. on the recovery factor if we accept Crocco's formula as a good

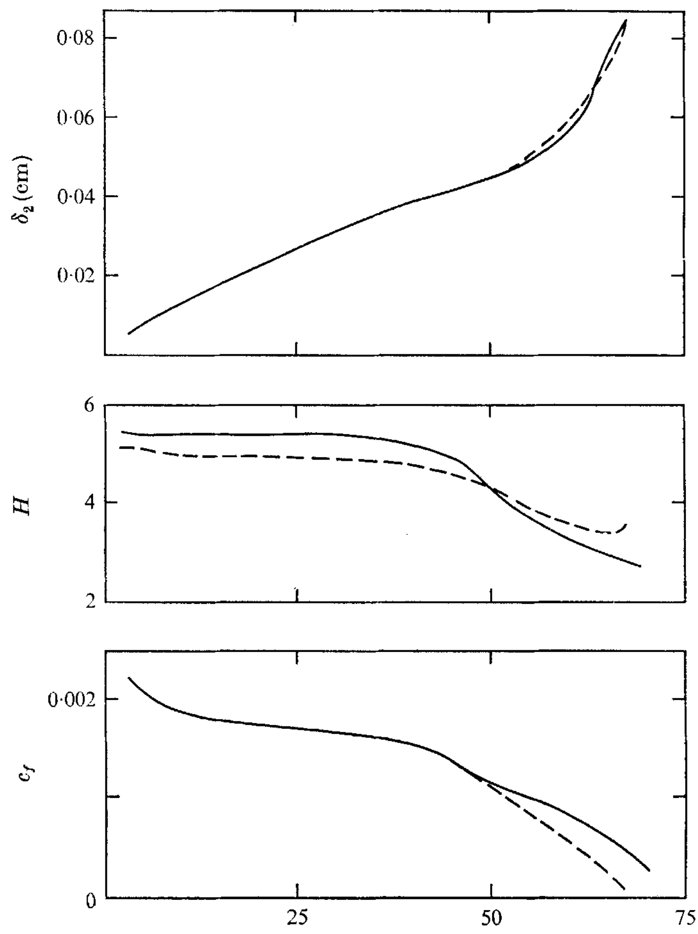

FIGURE 8. Fictitious separating boundary layer (with $M_1 = 3$ for $x < 45$ cm, decreasing linearly to $M_1 = 1{\cdot}0$ at $x = 75$ cm). – – – –, present calculation; —, mixing-length calculation (Sivasegaram 1970).

approximation). In the test cases presented here, Pasiuk *et al.* measured the temperature directly (obtaining the surprising result of a recovery factor greater than unity on some parts of a nominally adiabatic wall), Zwarts assumed constant total temperature ($r = 1$) in reducing his data, and Sivasegaram used the Crocco formula with $r = 0{\cdot}93$ (being a typical value in his intermittent wind tunnel with a stagnation temperature rather less than room temperature). In our predictions of boundary-layer development we have used $r = 0{\cdot}89$ throughout: the approximate effect of taking $r = 1$ solely for calculating H is shown in the figures. As mentioned above the effect of r on the main part of the calculation (e.g. c_f) is fairly small. H_i is probably a more useful parameter for comparison but neither H nor H_i adds much to the comparison for c_f because H_i and c_f are coupled quite strongly by the logarithmic law for the inner layer.

If the flow is truly two-dimensional or if lateral convergence is allowed for, comparison between experimental and calculated momentum thickness adds nothing to comparisons of c_f and H because the three are connected by the momentum integral equation. In Zwarts's flow the difference between the experimental and calculated δ_2 is mainly due to inaccuracies in estimating the apparent lateral convergence from the imbalance in the experimental momentum integral equation: the large difference between experimental and calculated c_f has comparatively little effect.

Figure 8, reproduced from Bradshaw *et al.* (1970), shows a comparison of calculations by the present method and by the compressible version of the Patankar–Spalding mixing-length method for a fictitious but typical separating boundary layer. The difference in H, even at the initial station, results from differences in recovery factor (taken as 0·89 in the present method): the effect on c_f is negligible. It can be seen that the skin-friction predictions of the two methods diverge soon after the start of the adverse pressure gradient, and the distance from the start of the adverse gradient to the separation point (extrapolated for the mixing-length calculation) differs by 15 % between the two cases. The difference between the present method and the 'local-equilibrium' mixing-length method is likely to be significant in practice.

8. Conclusions

Although the data available for testing the present calculation method are not as reliable or extensive as one could wish, the results in zero pressure gradient seem very satisfactory, even at Mach number where $(\gamma-1)M_1^2$ is quite large instead of being of order unity as required by our cautious version of Morkovin's hypothesis. Agreement with experiment in strong pressure gradients at supersonic speeds is not very good but the data are suspect. Since the *incompressible* version of the method works well in a wide range of pressure gradients and the extra assumptions required for extension to compressible flow are not likely to be affected by pressure gradient, it is unlikely that the combination of compressibility and pressure gradient will seriously degrade the results.

As in incompressible flow, the assumptions are valid only for slowly-changing shear layers (i.e. those obeying the boundary-layer approximation): they would not be valid, for instance, in or just downstream of a shock-wave/boundary-layer interaction.

It seems that in practice compressible boundary layers may often suffer from the effects of low Reynolds number or large surface curvature. While neither of these effects is fully understood the present calculation method is soundly enough based for their occurrence, or even their magnitude, to be predited approximately.

Algol or Fortran listings for the computer program are available from the first author.

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